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Harmonic morphisms from the compact semisimple Lie groups and their non-compact duals

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Abstract

In this paper we prove the local existence of complex-valued harmonic morphisms from any compact semisimple Lie group and their non-compact duals. These include all Riemannian symmetric spaces of types II and IV. We produce a variety of concrete harmonic morphisms from the classical compact simple Lie groups $\mathbf{SO}(n)$, $\mathbf{SU}(n)$, $\mathbf{Sp}(n)$ and globally defined solutions on their non-compact duals $\mathbf{SO}(n,\mathbb{C})/\mathbf{SO}(n)$, $\mathbf{SL}_n(\mathbb{C})/\mathbf{SU}(n)$ and $\mathbf{Sp}(n,\mathbb{C})/\mathbf{Sp}(n)$.

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1. Introduction

Harmonic morphisms between Riemannian manifolds are harmonic maps which satisfy an additional conformality condition, called horizontal (weak) conformality. As their fibres are often minimal, they are useful tools for the construction of minimal submanifolds. However, the two conditions imposed on them constitute an over-determined non-linear system of partial differential equations. For this reason they can be difficult to find and have no general existence theory, not even locally. On the contrary, most metrics on a 3-dimensional domain M^3 do not allow any local solutions with values in a surface N^2, see [5]. This makes it interesting to find geometric and topological conditions on the manifolds (M,g) and (N,h) which ensure the existence of solutions to the problem. For the general theory of harmonic morphisms between Riemannian manifolds we refer to the excellent book [4] and the regularly updated on-line bibliography [7].

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For harmonic morphisms $\phi:(M,g)\to(N,h)$ to exist, it is an advantage if the target manifold N is a surface, i.e. of dimension 2. In this case the problem is invariant under conformal changes of the metric on N^2. Hence, at least for local studies, the codomain can be assumed to be the standard complex plane.

It is known that in several cases, when the domain (M,g) is an irreducible Riemannian symmetric space, there exist complex-valued solutions to the problem, see for example [8,9,13], where the authors present the following conjecture.

Conjecture 1.1. *Let (M^m,g) be an irreducible Riemannian symmetric space of dimension $m\geqslant 2$. For each point $p\in M$ there exists a complex-valued harmonic morphism $\phi:U\to\mathbb{C}$ defined on an open neighbourhood U of p. If the space (M,g) is of non-compact type then the domain U can be chosen to be the whole of M.*

The conjecture is known to be true for the irreducible Riemannian symmetric spaces

$$\mathbf{SO}(p+q)/\mathbf{SO}(p)\times\mathbf{SO}(q),\qquad \mathbf{SO}_0(p,q)/\mathbf{SO}(p)\times\mathbf{SO}(q)$$

when $p\notin\{q,q\pm1\}$,

$$\mathbf{SU}(p+q)/\mathbf{S}\big(\mathbf{U}(p)\times\mathbf{U}(q)\big),\qquad \mathbf{SU}(p,q)/\mathbf{S}\big(\mathbf{U}(p)\times\mathbf{U}(q)\big)$$

for any positive integers p,q, and for

$$\mathbf{Sp}(p+q)/\mathbf{Sp}(p)\times\mathbf{Sp}(q),\qquad \mathbf{Sp}(p,q)/\mathbf{Sp}(p)\times\mathbf{Sp}(q)$$

when $p\neq q$, see [9].

In this paper we continue our study of complex-valued harmonic morphisms from Riemannian symmetric spaces. We prove the following existence theorem, see Section 9. For the definition of an orthogonal harmonic family, see the next section.

Theorem 1.2. *Let G be a compact semisimple Lie group equipped with a bi-invariant metric g and let $G^{\mathbb{C}}/G$ be its non-compact dual space.*

(i) *There exists an open and dense subset W^* of G and an orthogonal harmonic family $\mathcal{F}^*$ on W^*.*
(ii) *There exists an open subset W of $G^{\mathbb{C}}/G$ and an orthogonal harmonic family $\mathcal{F}$ on W.*
(iii) *If there exists a parabolic subgroup P of $G^{\mathbb{C}}$ such that the quotient $G^{\mathbb{C}}/P$ is a Hermitian symmetric space, then there is a globally defined orthogonal harmonic family $\mathcal{F}$ on $G^{\mathbb{C}}/G$.*

The collections of compact semisimple Lie groups and their non-compact duals include the irreducible Riemannian symmetric spaces of type II and IV, respectively. This means that by Theorem 1.2 we prove Conjecture 1.1 for the compact irreducible Riemannian symmetric spaces

$$\mathbf{SO}(n),\qquad \mathbf{SU}(n),\qquad \mathbf{Sp}(n),\qquad E_6,\qquad E_7,\qquad E_8,\qquad F_4,\qquad G_2$$

of type II, and the non-compact

$$\mathbf{SO}(n,\mathbb{C})/\mathbf{SO}(n),\qquad \mathbf{SL}_n(\mathbb{C})/\mathbf{SU}(n),\qquad \mathbf{Sp}(n,\mathbb{C})/\mathbf{Sp}(n),\qquad E_6^{\mathbb{C}}/E_6,\qquad E_7^{\mathbb{C}}/E_7$$

of type IV.

Leading up to the general existence theory we produce a variety of *concrete* complex-valued harmonic morphisms on the irreducible Riemannian symmetric spaces $\mathbf{SO}(n)$, $\mathbf{SU}(n)$, $\mathbf{Sp}(n)$ of type II and $\mathbf{SO}(n,\mathbb{C})/\mathbf{SO}(n)$, $\mathbf{SL}_n(\mathbb{C})/\mathbf{SU}(n)$, $\mathbf{Sp}(n,\mathbb{C})/\mathbf{Sp}(n)$ of type IV.

Throughout this article we assume that all our manifolds are connected. Further we suppose that all our objects such as manifolds, maps etc. are smooth, i.e. in the C^∞-category. For our notation concerning Lie groups we refer to the comprehensive book [12].

2. Harmonic morphisms

Let M and N be two manifolds of dimensions m and n, respectively. A Riemannian metric g on M gives rise to the notion of a *Laplacian* on (M,g) and real-valued *harmonic functions* $f:(M,g)\to\mathbb{R}$. This can be generalized to the concept of *harmonic maps* $\phi:(M,g)\to(N,h)$ between Riemannian manifolds, which are solutions to a semi-linear system of partial differential equations, see [4].

Definition 2.1. A map $\phi:(M,g)\to(N,h)$ between Riemannian manifolds is called a *harmonic morphism* if, for any harmonic function $f:U\to\mathbb{R}$ defined on an open subset U of N with $\phi^{-1}(U)$ non-empty, $f\circ\phi:\phi^{-1}(U)\to\mathbb{R}$ is a harmonic function.

The following characterization of harmonic morphisms between Riemannian manifolds is due to Fuglede and Ishihara. For the definition of horizontal (weak) conformality we refer to [4].

Theorem 2.2. [6,11] *A map $\phi:(M,g)\to(N,h)$ between Riemannian manifolds is a harmonic morphism if and only if it is a horizontally* (*weakly*) *conformal harmonic map.*

The following result of Baird and Eells gives the theory of harmonic morphisms a strong geometric flavour and shows that the case when $n=2$ is particularly interesting. The conditions characterizing harmonic morphisms are then independent of conformal changes of the metric on the surface N^2. For the definition of horizontal homothety we refer to [4].

Theorem 2.3. [3] *Let $\phi:(M^m,g)\to(N^n,h)$ be a horizontally conformal submersion between Riemannian manifolds. If*

(i) $n=2$*, then ϕ is harmonic if and only if ϕ has minimal fibres,*
(ii) $n\geqslant 3$*, then two of the following conditions imply the other*:
  (a) *ϕ is a harmonic map,*
  (b) *ϕ has minimal fibres,*
  (c) *ϕ is horizontally homothetic.*

In this article we are interested in complex-valued functions $\phi,\psi:(M,g)\to\mathbb{C}$ from Riemannian manifolds. In this situation the metric g induces the complex-valued Laplacian $\tau(\phi)$ and the gradient $\operatorname{grad}(\phi)$ with values in the complexified tangent bundle $T^{\mathbb{C}}M$ of M. We extend the metric g to be complex bilinear on $T^{\mathbb{C}}M$ and define the symmetric bilinear operator κ by

$$\kappa(\phi,\psi)=g\big(\operatorname{grad}(\phi),\operatorname{grad}(\psi)\big).$$

Two maps $\phi,\psi:M\to\mathbb{C}$ are said to be *orthogonal* if

$$\kappa(\phi,\psi)=0.$$

The harmonicity and horizontal conformality of $\phi:(M,g)\to\mathbb{C}$ are expressed by the relations

$$\tau(\phi)=0\quad\text{and}\quad\kappa(\phi,\phi)=0.$$

Definition 2.4. Let (M,g) be a Riemannian manifold. Then a set

$$\Omega=\{\phi_k:M\to\mathbb{C}\mid k\in I\}$$

of complex-valued functions is said to be an *orthogonal harmonic family* on M if, for all $\phi,\psi\in\Omega$,

$$\tau(\phi)=0\quad\text{and}\quad\kappa(\phi,\psi)=0.$$

Remark 2.5. For a finite orthogonal harmonic family $\{\phi_1, \dots, \phi_k\}$ on a Riemannian manifold (M, g), the map

$$\Phi = (\phi_1, \dots, \phi_k) : M \to \mathbb{C}^k$$

is a *pseudo horizontally* (*weakly*) *conformal* map. See for example [4, Definition 8.2.3 and Example 8.2.6].

We have the following important example of orthogonal harmonic families.

Example 2.6. Let (M, g, J) be a Kähler manifold and

$$\Omega = \{f_k : M \to \mathbb{C} \mid k \in I\}$$

be a collection of holomorphic functions on M. For $f \in \Omega$, the gradient $\text{grad}(f)$ satisfies

$$g\big(J\,\text{grad}(f), X\big) = -g\big(\text{grad}(f), JX\big) = -df(JX) = -i\,df(X) = g\big(-i\,\text{grad}(f), X\big).$$

This implies that the gradients $\text{grad}(f_k)$ all belong to the subbundle $T^{0,1}M$ of the complexified tangent bundle $T^{\mathbb{C}}M$ of M. Since $T^{0,1}M$ is isotropic, i.e. $g(Z, W) = 0$ for all $Z, W \in T^{0,1}M$, and any holomorphic complex-valued function on a Kähler manifold is harmonic, we see that Ω is an orthogonal harmonic family on M.

The next result shows that an orthogonal harmonic family on a Riemannian manifold can be used to produce a variety of harmonic morphisms.

Theorem 2.7. [9] *Let (M, g) be a Riemannian manifold and*

$$\Omega = \{\phi_k : M \to \mathbb{C} \mid k = 1, \dots, n\}$$

be a finite orthogonal harmonic family on (M, g). Let $\Phi : M \to \mathbb{C}^n$ be the map given by $\Phi = (\phi_1, \dots, \phi_n)$ and U be an open subset of $\mathbb{C}^n$ containing the image $\Phi(M)$ of Φ. If

$$\tilde{\mathcal{F}} = \{F_i : U \to \mathbb{C} \mid i \in I\}$$

is a family of holomorphic functions then

$$\mathcal{F} = \{\psi : M \to \mathbb{C} \mid \psi = F(\phi_1, \dots, \phi_n),\ F \in \tilde{\mathcal{F}}\}$$

is an orthogonal harmonic family on M.

The main aim of this paper is to construct orthogonal harmonic families on the Riemannian symmetric spaces that we are dealing with and thereby prove Conjecture 1.1 in those cases.

3. The classical compact cases

For a compact Riemannian symmetric space G/K the natural projection $\pi : G \to G/K$ is a harmonic morphism. Thus the lift, of any harmonic morphism locally defined on G/K, to the Lie group G via π is a harmonic morphism. This means that for positive integers p, q and $n = p + q$, complex-valued harmonic morphisms defined locally on the real, complex or quaternionic Grassmannians

$$\mathbf{SO}(n)/\mathbf{SO}(p) \times \mathbf{SO}(q), \qquad \mathbf{SU}(n)/\mathbf{S}\big(\mathbf{U}(p) \times \mathbf{U}(q)\big), \qquad \mathbf{Sp}(n)/\mathbf{Sp}(p) \times \mathbf{Sp}(q),$$

can be lifted to local harmonic morphisms on the corresponding compact Lie group $\mathbf{SO}(n)$, $\mathbf{SU}(n)$ or $\mathbf{Sp}(n)$. For later use, we now describe explicitly how this works for the harmonic morphisms presented in [9].

Example 3.1. Let p, q be positive integers and $W^*_{pq}(\mathbb{C})$ be the open subset of the special unitary group $\mathbf{SU}(p+q)$ given by

$$W^*_{pq}(\mathbb{C}) = \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathbf{SU}(p+q) \,\middle|\, \det(x_{11}) \neq 0 \right\}$$

where $x_{11} \in \mathbb{C}^{p\times p}$. Then the complex-valued components of the map

$$\Phi^* : W^*_{pq}(\mathbb{C}) \to \mathbb{C}^{q\times p}, \qquad \Phi^* : x \mapsto x_{21} \cdot x_{11}^{-1}$$

form an orthogonal harmonic family on $W^*_{pq}(\mathbb{C})$.

Example 3.2. Let p, r be positive integers $q = p + 2r$ and $W^*_{pq}(\mathbb{R})$ be the open subset of the special orthogonal group $\mathbf{SO}(p+q)$ given by

$$W^*_{pq}(\mathbb{R}) = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix} \in \mathbf{SO}(p+q) \,\middle|\, \det(x_{11} + i x_{21}) \neq 0 \right\}$$

where $x_{11}, x_{21} \in \mathbb{R}^{p\times p}$. Then the complex-valued components of the map

$$\Phi^* : W^*_{pq}(\mathbb{R}) \to \mathbb{C}^{r\times p}, \qquad \Phi^* : x \mapsto (x_{31} + i x_{41}) \cdot (x_{11} + i x_{21})^{-1}$$

form an orthogonal harmonic family on $W^*_{pq}(\mathbb{R})$.

By employing further examples from [9] one can actually construct locally defined complex-valued harmonic morphisms on $\mathbf{SO}(n)$ not only for n even but for any $n > 2$. This we leave to the reader as an exercise.

Example 3.3. Recall that the Lie groups $\mathbf{Sp}(n, \mathbb{C})$ and $\mathbf{Sp}(n)$ are given by

$$\mathbf{Sp}(n, \mathbb{C}) = \{ g \in \mathbf{SL}_{2n}(\mathbb{C}) \mid g^t J g = J \},$$
$$\mathbf{Sp}(n) = \mathbf{Sp}(n, \mathbb{C}) \cap \mathbf{U}(2n) = \{ g \in \mathbf{SU}(2n) \mid g J = J \bar{g} \},$$

where I_n denotes the $n \times n$ identity matrix and

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Let p, r be positive integers, $q = p + r$, and let $W^*_{pq}(\mathbb{H})$ be the open subset of $\mathbf{Sp}(p + p + r)$ consisting of elements

$$x = \begin{pmatrix} z_{11} & z_{12} & z_{13} & w_{11} & w_{12} & w_{13} \\ z_{21} & z_{22} & z_{23} & w_{21} & w_{22} & w_{23} \\ z_{31} & z_{32} & z_{33} & w_{31} & w_{32} & w_{33} \\ -\bar{w}_{11} & -\bar{w}_{12} & -\bar{w}_{13} & \bar{z}_{11} & \bar{z}_{12} & \bar{z}_{13} \\ -\bar{w}_{21} & -\bar{w}_{22} & -\bar{w}_{23} & \bar{z}_{21} & \bar{z}_{22} & \bar{z}_{23} \\ -\bar{w}_{31} & -\bar{w}_{32} & -\bar{w}_{33} & \bar{z}_{31} & \bar{z}_{32} & \bar{z}_{33} \end{pmatrix}$$

such that the matrix

$$\begin{pmatrix} z_{11} - z_{21} & w_{21} - w_{11} \\ \bar{w}_{21} + \bar{w}_{11} & \bar{z}_{11} + \bar{z}_{21} \end{pmatrix} \in \mathbb{C}^{2p\times 2p}$$

is invertible. Define the map $\Phi^* : W^*_{pq}(\mathbb{H}) \to \mathbb{C}^{r\times 2p}$ by

$$\Phi^* : x \mapsto \begin{pmatrix} z_{31} & -w_{31} \end{pmatrix} \begin{pmatrix} z_{11} - z_{21} & w_{21} - w_{11} \\ \bar{w}_{21} + \bar{w}_{11} & \bar{z}_{11} + \bar{z}_{21} \end{pmatrix}^{-1}.$$

Then the complex-valued components of Φ^* constitute an orthogonal harmonic family on $W^*_{pq}(\mathbb{H})$.

It is well known that the symmetric spaces $\mathbf{Sp}(2)/\mathbf{Sp}(1) \times \mathbf{Sp}(1)$ and $\mathbf{SO}(5)/\mathbf{SO}(1) \times \mathbf{SO}(4)$ are isometric. This fact together with the examples presented above, show that we have proved Conjecture 1.1 whenever the irreducible Riemannian symmetric space (M, g) is one of the classical compact Lie groups $\mathbf{SO}(n)$, $\mathbf{SU}(n)$ or $\mathbf{Sp}(n)$.

4. The duality

In this section we show how an orthogonal harmonic family on a compact semisimple Lie group gives rise to an orthogonal harmonic family on its non-compact dual space. Let G be a compact semisimple Lie group with a bi-invariant Riemannian metric and let $\mathfrak{g}$ be its Lie algebra. For the induced Lie groups

$$U = G \times G \quad \text{and} \quad K = \{(g, g) \in U \mid g \in G\}$$

we have the isomorphism $U/K \cong G$ given by

$$(g_1, g_2)K \mapsto g_1 g_2^{-1}.$$

As K is the fixed point set of the involutive automorphism $\sigma^* : U \to U$ with

$$\sigma^* : (g_1, g_2) \mapsto (g_2, g_1),$$

we see that $G \cong U/K$ is a Riemannian globally symmetric space. The Lie algebra of U decomposes

$$\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{p}$$

into the ± 1 eigenspaces of the differential $s^* = (d\sigma^*)_e : \mathfrak{u} \to \mathfrak{u}$ of σ^* at e with

$$\mathfrak{k} = \{(X, X) \mid X \in \mathfrak{g}\}, \qquad \mathfrak{p} = \{(Y, -Y) \mid Y \in \mathfrak{g}\},$$

and $\mathfrak{k}$ is the Lie algebra of K.

The complexification $\mathfrak{u}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{u}$ of $U = G \times G$ satisfies

$$\mathfrak{u}^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}.$$

Let $s^{\mathbb{C}} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ be the involutive automorphism on the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ with $s^{\mathbb{C}} : Z \mapsto \bar{Z}$, where the conjugation is done according to the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}.$$

Furthermore, denote by $\sigma^{\mathbb{C}} : G^{\mathbb{C}} \to G^{\mathbb{C}}$ the involutive automorphism on the complexification $G^{\mathbb{C}}$ of G induced by $s^{\mathbb{C}}$. Let $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{u}^{\mathbb{C}}$ given by

$$\mathfrak{h} = \mathfrak{k} + i\mathfrak{p} = \{(X, X) + i(Y, -Y) \mid X, Y \in \mathfrak{g}\} = \{(Z, \bar{Z}) \mid Z \in \mathfrak{g}^{\mathbb{C}}\}.$$

Let $s : \mathfrak{h} \to \mathfrak{h}$ be the involutive automorphism on $\mathfrak{h}$ with

$$s(Z, \bar{Z}) = (\bar{Z}, Z).$$

Then the Lie algebra $\mathfrak{k}$ of K is the fixed point set of s and $(\mathfrak{h}, s)$ is an orthogonal symmetric Lie algebra which is dual to $(\mathfrak{u}, s^*)$. A Lie group with Lie algebra $\mathfrak{h}$ is given by

$$H = \{(g, \sigma^{\mathbb{C}}(g)) \mid g \in G^{\mathbb{C}}\}.$$

The automorphism s on $\mathfrak{h}$ gives an automorphism σ on H with fixed point set K. Hence (H, K) is a Riemannian symmetric pair which is dual to (U, K). Note that we have an isomorphism $H \cong G^{\mathbb{C}}$ with

$$(g, \sigma^{\mathbb{C}}(g)) \mapsto g.$$

Under this isomorphism we have $K \cong G$ and σ corresponds to $\sigma^{\mathbb{C}}$. This shows that the quotient $G^{\mathbb{C}}/G$ is isometric to H/K and hence it is the dual space of $G \cong U/K$.

Let $\phi^* : G \to \mathbb{C}$ be a real analytic map. We can extend ϕ^* uniquely to a holomorphic map $\phi^{\mathbb{C}} : G^{\mathbb{C}} \to \mathbb{C}$, assuming for simplicity that this is defined on all of $G^{\mathbb{C}}$. We lift ϕ^* to

$$\hat{\phi}^* : U = G \times G \to \mathbb{C},$$

via the natural projection $\pi^* : U \to U/K$, and uniquely extend $\hat{\phi}^*$ to a holomorphic map

$$\hat{\phi}^{\mathbb{C}} : U^{\mathbb{C}} = G^{\mathbb{C}} \times G^{\mathbb{C}} \to \mathbb{C}.$$

Then the restriction $\hat{\phi} : H \to \mathbb{C}$ of $\hat{\phi}^{\mathbb{C}}$ to H is given by

$$\hat{\phi}(h) = \hat{\phi}^{\mathbb{C}}\big(h, \sigma^{\mathbb{C}}(h)\big) = \phi^{\mathbb{C}}\big(h \cdot \big(\sigma^{\mathbb{C}}(h)\big)^{-1}\big),$$

where we have used the identification $H \cong G^{\mathbb{C}}$. As this map is K-invariant, it induces a map $\phi : H/K \to \mathbb{C}$ from the dual space H/K of U/K, given by

$$\phi(hK) = \phi^{\mathbb{C}}\big(h \cdot \big(\sigma^{\mathbb{C}}(h)\big)^{-1}\big).$$

The construction above can of course be restricted to open subsets W^* and W of U/K and H/K, respectively. For that situation we have the following useful duality theorem.

Theorem 4.1. [9] *Let $\mathcal{F}$ be a family of maps $\phi : W \to \mathbb{C}$ and $\mathcal{F}^*$ be the dual family consisting of the maps $\phi^* : W^* \to \mathbb{C}$ constructed as above. Then $\mathcal{F}^*$ is an orthogonal harmonic family on W^* if and only if $\mathcal{F}$ is an orthogonal harmonic family on W.*

5. Global solutions on $\mathbf{SL}_n(\mathbb{C})/\mathbf{SU}(n)$

Let p, q be positive integers and $n = p + q$. The complexification of the special unitary group $\mathbf{SU}(n)$ is the special linear group $\mathbf{SL}_n(\mathbb{C})$, which clearly is a complex submanifold of the general linear group $\mathbf{GL}_n(\mathbb{C})$. In Example 3.1 we have constructed the map $\Phi^* : W^*_{pq}(\mathbb{C}) \to \mathbb{C}^{q\times p}$ defined on an open subset of $\mathbf{SU}(n)$. This extends analytically to the holomorphic map

$$\Phi^{\mathbb{C}} : W^{\mathbb{C}}_{pq}(\mathbb{C}) \to \mathbb{C}^{q\times p}, \qquad \Phi^{\mathbb{C}} : z \mapsto z_{21} \cdot z_{11}^{-1}$$

defined on the following open subset of $\mathbf{SL}_n(\mathbb{C})$:

$$W^{\mathbb{C}}_{pq}(\mathbb{C}) = \left\{ \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathbf{SL}_{p+q}(\mathbb{C}) \,\middle|\, \det(z_{11}) \neq 0 \right\}.$$

The involutive automorphism $s^{\mathbb{C}} : \mathfrak{sl}_n(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C})$ on the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ is given by the conjugation

$$s^{\mathbb{C}} : Z + iW \mapsto Z - iW = -(Z + iW)^*$$

with respect to the decomposition $\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{su}(n) \oplus i\mathfrak{su}(n)$. This implies that, for the involutive automorphism $\sigma^{\mathbb{C}}$ on $\mathbf{SL}_n(\mathbb{C})$ induced by $s^{\mathbb{C}}$, we have

$$\sigma^{\mathbb{C}}(z) = (z^*)^{-1}.$$

We now get a globally defined $\mathbf{SU}(n)$-invariant map $\Phi : \mathbf{SL}_n(\mathbb{C}) \to \mathbb{C}^{q\times p}$ defined by

$$\Phi(z) = \Phi^{\mathbb{C}}\big(z \cdot \big(\sigma^{\mathbb{C}}(z)^{-1}\big)\big) = \Phi^{\mathbb{C}}(z \cdot z^*) = (z_{21}z_{11}^* + z_{22}z_{12}^*)(z_{11}z_{11}^* + z_{12}z_{12}^*)^{-1}.$$

According to Theorem 4.1 the complex-valued components of the map Φ constitute an orthogonal harmonic family on the irreducible Riemannian symmetric space $\mathbf{SL}_n(\mathbb{C})/\mathbf{SU}(n)$.

6. Global solutions on $\mathbf{SO}(n, \mathbb{C})/\mathbf{SO}(n)$

In this section we apply the Hermitian structure on the symmetric space $\mathbf{SO}_0(2, q)/\mathbf{SO}(2) \times \mathbf{SO}(q)$ to construct an orthogonal harmonic family on the irreducible Riemannian symmetric space

$$\mathbf{SO}(n, \mathbb{C})/\mathbf{SO}(n)$$

for $n \geqslant 3$, where $n = 2 + q$. Recall that the Hermitian symmetric space

$$\mathbf{SO}_0(2, q)/\mathbf{SO}(2) \times \mathbf{SO}(q)$$

is isomorphic to

$$\mathcal{D} = \big\{ X \in \mathbb{R}^{2\times q} \mid XX^t < I_2 \big\}$$

through the $\mathbf{SO}(2)\times\mathbf{SO}(q)$-invariant map

$$\mathbf{SO}_0(2,q)\ni\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\mapsto\beta\delta^{-1}\in\mathcal{D}.$$

For this see for example [10, p. 527]. The set $\mathcal{D}$ can be realized as a bounded symmetric domain in $\mathbb{C}^q$ through the map

$$\mathbb{R}^{2\times q}\ni X=\begin{pmatrix}X_1\\ X_2\end{pmatrix}\mapsto X_2+iX_1=X^{\mathbb{C}}\in\mathbb{C}^q,$$

where the last equality defines $X^{\mathbb{C}}$. Thus, the map

$$\mathbf{SO}_0(2,q)\ni\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\mapsto\beta^{\mathbb{C}}\delta^{-1}$$

realizes the space $\mathbf{SO}_0(2,q)/\mathbf{SO}(2)\times\mathbf{SO}(q)$ as a Hermitian symmetric space; its coordinates constitute a globally defined holomorphic functions, and hence a globally defined orthogonal harmonic family.

As in [9, Example 7.2], we introduce the group

$$G=\{g\in\mathbf{SO}(n,\mathbb{C})\mid g^*I_{2q}g=I_{2q}\},$$

where

$$I_{pq}=\begin{pmatrix}-I_p & 0\\ 0 & I_q\end{pmatrix}.$$

Recall that we have an $\mathbf{SO}(2)\times\mathbf{SO}(q)$-preserving isomorphism

$$G\ni\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\mapsto\begin{pmatrix}\alpha & -i\beta\\ i\gamma & \delta\end{pmatrix}\in\mathbf{SO}_0(2,q).$$

So, lifting the orthogonal harmonic family first to $\mathbf{SO}_0(2,q)$, composing it with this isomorphism from G and then extending this composition to the complexification $\mathbf{SO}(n,\mathbb{C})$, gives the map

$$\Phi^{\mathbb{C}}:\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\mathbf{SO}(2+q,\mathbb{C})\,\middle|\,\det\delta\neq 0\right\}\mapsto -i\beta^{\mathbb{C}}\delta^{-1}\in\mathbb{C}^q.$$

Conjugation in $\mathfrak{so}(n,\mathbb{C})$ with respect to the compact real form $\mathfrak{so}(n)$ is given by

$$s^{\mathbb{C}}(X)=-X^*,$$

and the induced involutive automorphism of $\mathbf{SO}(n,\mathbb{C})$ satisfies

$$\sigma^{\mathbb{C}}(g)=(g^*)^{-1}.$$

We compose Φ with the map

$$\mathbf{SO}(n,\mathbb{C})\ni g\mapsto gg^*\in\mathbf{SO}(n,\mathbb{C}),$$

which, according to Theorem 4.1, produces an orthogonal harmonic family on some open subset of $\mathbf{SO}(n,\mathbb{C})/\mathbf{SO}(n)$. Explicitly, the map is given by

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\mapsto -i(\alpha\gamma^*+\beta\delta^*)^{\mathbb{C}}(\gamma\gamma^*+\delta\delta^*)^{-1}\in\mathbb{C}^q,$$

and is clearly globally defined.

7. Further global solutions on $\mathbf{SO}(2n,\mathbb{C})/\mathbf{SO}(2n)$

In this section we use the Hermitian structure on the symmetric space $\mathbf{SO}^*(2n)/\mathbf{U}(n)$ to construct an orthogonal harmonic family on the irreducible Riemannian symmetric space $\mathbf{SO}(2n,\mathbb{C})/\mathbf{SO}(2n)$. This is different from the one produced in the previous section, and its construction involves methods which will be used again later on.

The Lie group

$$\mathbf{SO}^*(2n)=\{g\in\mathbf{SO}(2n,\mathbb{C})\mid g^*Jg=J\}$$

is a real form of

$$\mathbf{SO}(2n,\mathbb{C})=\{g\in\mathbf{SL}_{2n}(\mathbb{C})\mid g^tg=I_n\},$$

and the image of the unitary group $\mathbf{U}(n)$ under the real representation

$$A+Bi\mapsto\begin{pmatrix}A & -B\\ B & A\end{pmatrix}$$

is embedded in $\mathbf{SO}^*(2n)$ as the fixed point set of the Cartan involution $\theta:\mathbf{SO}^*(2n)\to\mathbf{SO}^*(2n)$ with $\theta(g)=\bar{g}$. Let $\mathbf{SU}(n,n)$ be the non-compact Lie group defined by

$$\mathbf{SU}(n,n)=\{g\in\mathbf{SL}_{2n}(\mathbb{C})\mid g^*I_{nn}g=I_{nn}\}$$

and G be the subgroup of $\mathbf{SU}(n,n)$ with

$$G=\{g\in\mathbf{SU}(n,n)\mid g^tI_{nn}Jg=I_{nn}J\}.$$

Then the unitary group $\mathbf{U}(n)$ is embedded into G by

$$A\mapsto\begin{pmatrix}A & 0\\ 0 & \bar{A}\end{pmatrix}.$$

Introduce the matrix

$$K_n=\frac{1}{\sqrt{2}}\begin{pmatrix}-iI_n & I_n\\ iI_n & I_n\end{pmatrix}.$$

Proposition 7.1. *Define the map $\psi:\mathbf{SO}^*(2n)\to G$ by conjugation with K_n*

$$\psi:g\mapsto K_ngK_n^{-1}.$$

Then ψ is a $\mathbf{U}(n)$-preserving group isomorphism.

Proof. First, note that the matrix K_n is unitary so $K_n^*=K_n^{-1}$. Moreover,

$$K_nK_n^t=-I_{nn}J,\qquad K_n^*I_{nn}K_n=-iJ,\qquad K_n^tI_{nn}JK_n=-I_{2n}.$$

This means that if $g\in\mathbf{SO}^*(2n)$ then

$$(K_ngK_n^*)^*I_{nn}K_ngK_n^*=K_ng^*K_n^*I_{nn}K_ngK_n^*=-iK_ng^*JgK_n^*=-iK_nJK_n^*=I_{nn}$$

and

$$(K_ngK_n^*)^tI_{nn}JK_ngK_n^*=\bar{K}_ng^tK_n^tI_{nn}JK_ngK_n^*=-\bar{K}_ng^tgK_n^*=-\bar{K}_nK_n^*=-\overline{K_nK_n^t}=I_{nn}J.$$

This shows that the conjugation by K_n defines an isomorphism between the groups $\mathbf{SO}^*(2n)$ and G. Finally we see that

$$K_n\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}K_n^*=\begin{pmatrix}\alpha+i\beta & 0\\ 0 & \alpha-i\beta\end{pmatrix},$$

so ψ preserves the unitary group $\mathbf{U}(n)$. $\square$

It is well known that the non-compact irreducible Hermitian symmetric space $G/\mathbf{U}(n)$ can be represented as the bounded symmetric domain

$$\mathcal{D} = \{Z \in \mathbb{C}^{n\times n} \mid I_n - ZZ^* > 0,\ Z^t = -Z\}$$

and that G acts on $\mathcal{D}$ as a group of isometries by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \mapsto (AZ + B)(CZ + D)^{-1}.$$

This means that the natural projection $\pi : G \to \mathcal{D}$ is given by

$$\pi : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto B \cdot D^{-1}$$

and that the $\mathbf{U}(n)$-invariant map $\tilde{\Phi} = \pi \circ \psi : \mathbf{SO}^*(2n) \to \mathcal{D}$ with

$$\tilde{\Phi} : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto (-\alpha - i\gamma - i\beta + \delta)(\alpha - i\gamma + i\beta + \delta)^{-1}$$

provides us with a representation of $\mathbf{SO}^*(2n)/\mathbf{U}(n)$ as the bounded symmetric domain $\mathcal{D}$.

Theorem 7.2. *Let $\tilde{\phi}_1, \dots, \tilde{\phi}_m : \mathbf{SO}^*(2n) \to \mathbb{C}$ be the complex-valued components of $\tilde{\Phi} : \mathbf{SO}^*(2n) \to \mathcal{D}$. Then the set*

$$\tilde{\Omega} = \{\tilde{\phi}_1, \dots, \tilde{\phi}_m\}$$

is an orthogonal harmonic family of $\mathbf{U}(n)$-invariant complex-valued functions on $\mathbf{SO}^*(2n)$.

Proof. The statement is a direct consequence of the fact that the coordinate functions on the bounded symmetric domain $\mathcal{D}$ are holomorphic and therefore, by Example 2.6, they form an orthogonal harmonic family on $\mathcal{D}$. □

Let $\Phi^{\mathbb{C}} : W^{\mathbb{C}} \to \mathbb{C}^{n\times n}$ be the analytic extension of $\tilde{\Phi} : \mathbf{SO}^*(2n) \to \mathcal{D}$ to some open subset $W^{\mathbb{C}}$ of $\mathbf{SO}(2n, \mathbb{C})$ and $\Phi : W \to \mathcal{D}$ be the composition of $\Phi^{\mathbb{C}}$ with the map

$$\mathbf{SO}(2n, \mathbb{C}) \to \mathbf{SO}(2n, \mathbb{C}) \quad \text{with } g \mapsto gg^*.$$

According to Theorem 4.1 the components of Φ form an orthogonal harmonic family of $\mathbf{SO}(2n)$-invariant functions on the open subset W of $\mathbf{SO}(2n, \mathbb{C})$. The map $\Phi : W \to \mathbb{C}^{n\times n}$ is defined at a point $g \in \mathbf{SO}(2n, \mathbb{C})$ unless

$$\det\big(i(\gamma\alpha^* + \delta\beta^*) - i(\alpha\gamma^* + \beta\delta^*) + \gamma\gamma^* + \alpha\alpha^* + \beta\beta^* + \delta\delta^*\big) = 0.$$

In this case there exists a non-zero element $z \in \mathbb{C}^n$ such that

$$\begin{aligned} 0 &= i(z\gamma, z\alpha) + i(z\delta, z\beta) - i(z\alpha, z\gamma) - i(z\beta, z\delta) + |z\delta|^2 + |z\alpha|^2 + |z\beta|^2 + |z\delta|^2 \\ &= (iz\gamma, z\alpha) + (iz\delta, z\beta) + (z\alpha, iz\gamma) + (z\beta, iz\delta) + |iz\gamma|^2 + |z\alpha|^2 + |z\beta|^2 + |iz\delta|^2 \\ &= |z\alpha + iz\gamma|^2 + |z\beta + iz\delta|^2. \end{aligned}$$

But this means that

$$(z, iz)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 0,$$

which is impossible as the matrix is invertible. This shows that the function Φ is globally defined on $\mathbf{SO}(2n, \mathbb{C})$ and induces an orthogonal harmonic family on the irreducible Riemannian symmetric space

$$\mathbf{SO}(2n, \mathbb{C})/\mathbf{SO}(2n).$$

8. Global solutions on $\mathbf{Sp}(n,\mathbb{C})/\mathbf{Sp}(n)$

In this section we apply the Hermitian structure on the symmetric space $\mathbf{Sp}(n,\mathbb{R})/\mathbf{U}(n)$ to construct an orthogonal harmonic family on the irreducible Riemannian symmetric space $\mathbf{Sp}(n,\mathbb{C})/\mathbf{Sp}(n)$. The Lie group

$$\mathbf{Sp}(n,\mathbb{R}) = \{g \in \mathbf{SL}_{2n}(\mathbb{R}) \mid g^t J g = J\}$$

is the real form of

$$\mathbf{Sp}(n,\mathbb{C}) = \{g \in \mathbf{SL}_{2n}(\mathbb{C}) \mid g^t J g = J\}.$$

The image of the unitary group $\mathbf{U}(n)$ under the real representation

$$A + Bi \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

is embedded in $\mathbf{Sp}(n,\mathbb{R})$ as the fixed point set of the Cartan involution

$$\theta : \mathbf{Sp}(n,\mathbb{R}) \to \mathbf{Sp}(n,\mathbb{R}) \quad \text{with } g \mapsto J g J^t.$$

Let H be the subgroup of $\mathbf{SU}(n,n)$ given by

$$H = \{g \in \mathbf{SU}(n,n) \mid g^t J g = J\}.$$

Then the unitary group $\mathbf{U}(n)$ is embedded into H by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}.$$

As in the previous section, we introduce the matrix

$$K_n = \frac{1}{\sqrt{2}} \begin{pmatrix} -iI_n & I_n \\ iI_n & I_n \end{pmatrix}.$$

Proposition 8.1. *The map $\psi : \mathbf{Sp}(n,\mathbb{R}) \to H$ given by the conjugation with K_n*

$$\psi : g \mapsto K_n g K_n^{-1}$$

is a $\mathbf{U}(n)$-preserving group isomorphism.

Proof. The statement can be proved in a way similar to that of Proposition 7.1. $\square$

The Hermitian symmetric space $H/\mathbf{U}(n)$ can be represented as the bounded symmetric domain

$$\mathcal{D} = \{Z \in \mathbb{C}^{n\times n} \mid I_n - ZZ^* > 0,\ Z^t = Z\},$$

and H acts on $\mathcal{D}$ as a group of isometries by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \mapsto (AZ + B)(CZ + D)^{-1}.$$

The natural projection $\pi : H \to \mathcal{D}$ is given by

$$\pi : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto B \cdot D^{-1},$$

and the $\mathbf{U}(n)$-invariant map $\tilde{\Phi} = \pi \circ \psi : \mathbf{Sp}(n,\mathbb{R}) \to \mathcal{D}$ with

$$\tilde{\Phi} : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto (-\alpha - i\gamma - i\beta + \delta)(\alpha - i\gamma + i\beta + \delta)^{-1}$$

gives a representation of the Hermitian symmetric space $\mathbf{Sp}(n,\mathbb{R})/\mathbf{U}(n)$ as the bounded symmetric domain $\mathcal{D}$.

Theorem 8.2. *Let $\tilde{\phi}_1, \dots, \tilde{\phi}_m : \mathbf{Sp}(n, \mathbb{R}) \to \mathbb{C}$ be the complex valued components of $\tilde{\Phi} : \mathbf{Sp}(n, \mathbb{R}) \to \mathcal{D}$. Then the set*

$$\tilde{\Omega} = \{\tilde{\phi}_1, \dots, \tilde{\phi}_m\}$$

is an orthogonal harmonic family of $\mathbf{U}(n)$-invariant complex-valued functions on $\mathbf{Sp}(n, \mathbb{R})$.

Proof. The statement can be proved in the same way as that of Theorem 7.2. □

Let $\Phi^{\mathbb{C}} : W^{\mathbb{C}} \to \mathbb{C}^{n\times n}$ be the analytic extension of $\tilde{\Phi} : \mathbf{Sp}(n, \mathbb{R}) \to \mathcal{D}$ to some open subset $W^{\mathbb{C}}$ of $\mathbf{Sp}(n, \mathbb{C})$ and $\Phi : W \to \mathbb{C}^{n\times n}$ be the composition of $\Phi^{\mathbb{C}}$ with the map

$$\mathbf{Sp}(n, \mathbb{C}) \to \mathbf{Sp}(n, \mathbb{C}) \quad \text{with } g \mapsto gg^*.$$

Following Theorem 4.1 the components of Φ form an orthogonal harmonic family of $\mathbf{Sp}(n)$-invariant functions on the open subset W. Using the same argument as in the last section, it is easy to see that the map Φ is defined on the whole of $\mathbf{Sp}(n, \mathbb{C})$. Hence it induces an orthogonal harmonic family on the irreducible Riemannian symmetric space $\mathbf{Sp}(n, \mathbb{C})/\mathbf{Sp}(n)$.

9. General existence theory

In this section we establish a general existence theory for complex-valued harmonic morphisms from the compact semisimple Lie groups G and their non-compact dual spaces $G^{\mathbb{C}}/G$. These include the irreducible Riemannian symmetric spaces of type II and IV, respectively.

Let $\mathfrak{g}^{\mathbb{C}}$ be a complex, semisimple Lie algebra and B be its Killing form. Fix a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}$ and let

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha\in R} \mathfrak{g}^{\alpha}$$

be the corresponding root decomposition of $\mathfrak{g}^{\mathbb{C}}$. Here R denotes the set of (non-zero) roots. Let $\Pi \subset R$ be a basis of simple roots, inducing a decomposition $R = R^+ \cup R^-$ of R into positive and negative roots.

For each root $\alpha \in R$, let $H_\alpha \in \mathfrak{h}$ denote the dual element and let $\mathfrak{h}_{\mathbb{R}}$ be the real span of $\{H_\alpha \mid \alpha \in R\}$. Further choose elements $E_\alpha \in \mathfrak{g}^{\alpha}$ and $E_{-\alpha} \in \mathfrak{g}^{-\alpha}$, such that $[E_\alpha, E_{-\alpha}] = H_\alpha$. Then define the compact real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$ by

$$\mathfrak{g} = i\mathfrak{h}_{\mathbb{R}} \oplus \sum_{\alpha\in R^+} \mathbb{R}(E_\alpha + E_{-\alpha}) \oplus \sum_{\alpha\in R^+} \mathbb{R}i(E_\alpha - E_{-\alpha}),$$

and let τ denote the conjugation in $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$.

Let Π_K be any subset of Π and denote by R_K the set of roots which are linear combinations of elements in Π_K and by R_M the remaining roots. Introduce the subalgebras $\mathfrak{k}^{\mathbb{C}}$, $\mathfrak{p}_+$ and $\mathfrak{p}_-$ by

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha\in R_K} \mathfrak{g}^{\alpha}, \qquad \mathfrak{p}_+ = \sum_{\alpha\in R_M^+} \mathfrak{g}^{\alpha}, \qquad \mathfrak{p}_- = \sum_{\alpha\in R_M^+} \mathfrak{g}^{-\alpha}.$$

Then $\mathfrak{p} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}_+$ is a parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

Let $G^{\mathbb{C}}$ denote a complex, connected Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and let G, P_+, P_-, $K^{\mathbb{C}}$ and P denote the connected subgroups of $G^{\mathbb{C}}$ with the corresponding Lie algebras constructed above. Then the quotient

$$G^{\mathbb{C}}/P = G/(G \cap P)$$

is a (generalized) flag manifold, see [1,2]. Furthermore, we have

$$P = K^{\mathbb{C}} P_+.$$

For the Lie group G we have the natural reductive decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m},$$

where

$$\mathfrak{k} = i\mathfrak{h}_{\mathbb{R}} \oplus \sum_{\alpha \in R_K^+} \mathbb{R}(E_\alpha + E_{-\alpha}) \oplus \sum_{\alpha \in R_K^+} \mathbb{R}i(E_\alpha - E_{-\alpha}),$$

$$\mathfrak{m} = \sum_{\alpha \in R_M^+} \mathbb{R}(E_\alpha + E_{-\alpha}) \oplus \sum_{\alpha \in R_M^+} \mathbb{R}i(E_\alpha - E_{-\alpha}).$$

We can thus write the complexification $\mathfrak{m}^{\mathbb{C}}$ of $\mathfrak{m}$ as

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-.$$

This decomposition gives rise to a complex structure J on the generalized flag manifold

$$G^{\mathbb{C}}/P$$

for which $\mathfrak{p}_-$ is the $(1,0)$ space and $\mathfrak{p}_+$ the $(0,1)$ space at the identity coset. It is easy to see that the homogeneous projection

$$\pi^{\mathbb{C}} : G^{\mathbb{C}} \to G^{\mathbb{C}}/P$$

is holomorphic.

The Killing form B on the Lie algebra $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{m}$ induces a Hermitian metric $-B$ on the quotient

$$G/(P \cap G) = G^{\mathbb{C}}/P$$

and turns it into a cosymplectic manifold, see [14, Theorem 6.1]. Furthermore, the corresponding metric on G makes the homogeneous projection

$$\pi : G \to G/(P \cap G)$$

into a Riemannian submersion with totally geodesic fibres; in particular, a harmonic morphism.

The set $P_- P = P_- K^{\mathbb{C}} P_+$ is an open, dense subset of $G^{\mathbb{C}}$ and the map

$$\mathfrak{p}_- \ni X \mapsto \exp X \cdot P \in G^{\mathbb{C}}/P \tag{1}$$

is a holomorphic diffeomorphism onto an open, dense subset of $G^{\mathbb{C}}/P$, usually referred to as a *large cell*. For this we refer to [1].

To construct a harmonic orthogonal family on $G^{\mathbb{C}}/P$, we take complex coordinates on this large cell, i.e., the components of the inverse of the map (1). These complex-valued functions are holomorphic on a cosymplectic manifold, and so they do indeed form a harmonic orthogonal family on an open and dense subset of $G^{\mathbb{C}}/P = G/(G \cap P)$. We then lift this map using π to an open and dense subset of G. This proves part (i) of the following theorem.

Theorem 9.1. *Let G be a compact semisimple Lie group equipped with a bi-invariant metric g and let $G^{\mathbb{C}}/G$ be its non-compact dual space.*

(i) *There exists an open and dense subset W^* of G and an orthogonal harmonic family $\mathcal{F}^*$ on W^*.*
(ii) *There exists an open subset W of $G^{\mathbb{C}}/G$ and an orthogonal harmonic family $\mathcal{F}$ on W.*
(iii) *If there exists a parabolic subgroup P of $G^{\mathbb{C}}$ such that the quotient $G^{\mathbb{C}}/P$ is a Hermitian symmetric space, then there is a globally defined orthogonal harmonic family $\mathcal{F}$ on $G^{\mathbb{C}}/G$.*

Proof. (ii) First of all, we holomorphically extend the harmonic orthogonal family $\mathcal{F}$ on G to some open subset of $G^{\mathbb{C}}$. This is clearly the same as directly lifting the inverse of the map (1) to an open subset of $G^{\mathbb{C}}$ using the holomorphic projection $\pi^{\mathbb{C}}$. We then compose the lift with the map $G^{\mathbb{C}} \to G^{\mathbb{C}}$ given by

$$g \mapsto g\sigma^{\mathbb{C}}(g)^{-1},$$

where $\sigma^{\mathbb{C}} : G^{\mathbb{C}} \to G^{\mathbb{C}}$ is the involutive automorphism of G introduced in Section 4. By the duality Theorem 4.1 this induces a harmonic orthogonal family $\mathcal{F}$ on an open subset of the symmetric space $G^{\mathbb{C}}/G$. $\square$

Our final aim is to prove part (iii) of Theorem 9.1. This is clearly equivalent to the following result.

Lemma 9.2. *Let the quotient $G^{\mathbb{C}}/P$ be a Hermitian symmetric space. Then for each element g of the complex Lie group $G^{\mathbb{C}}$ the coset*

$$g\sigma^{\mathbb{C}}(g)^{-1}\cdot P$$

of $G^{\mathbb{C}}/P$ is contained in the corresponding large cell.

In order to prove Lemma 9.2 we recall that we may choose the Lie algebras $\mathfrak{p}_\pm$ to be Abelian and satisfy the relation

$$[\mathfrak{p}_-,\mathfrak{p}_+]\subseteq \mathfrak{k}^{\mathbb{C}}.$$

See for example [10, Chapter VIII].

Let V be an irreducible representation of $G^{\mathbb{C}}$ and let $v\in V$ be a non-zero vector of highest weight, such that the stabilizer of the line spanned by v in the projectivisation $\mathbb{P}V$ of V is the parabolic subgroup P of $G^{\mathbb{C}}$. Then

$$p\cdot v=\begin{cases} v & \text{if } p\in P_+,\\ \lambda(p)v \text{ for some } \lambda(p)\in\mathbb{C} & \text{if } p\in K^{\mathbb{C}}.\end{cases}$$

For a non-zero element $w\in V$, let $[w]$ denote the corresponding line in $\mathbb{P}V$ and let $\langle\,,\,\rangle$ be a fixed Hermitian inner product in V which is G-invariant.

Lemma 9.3. *For the above situation, we have the identity*

$$\bigl\{[w]\in G^{\mathbb{C}}\cdot[v]\mid \langle w,v\rangle\neq 0\bigr\}=P_-\cdot[v].$$

Proof of Lemma 9.3. If $[w]=p_-\cdot[v]$ for some $p_-\in P_-$, then there exists a complex number $\mu\neq 0$ such that

$$\langle w,v\rangle=\mu\langle p_-\cdot v,v\rangle=\mu\langle v,p_-^*\cdot v\rangle=\mu\bigl\langle v,\sigma^{\mathbb{C}}(p_-)^{-1}\cdot v\bigr\rangle=\mu|v|^2\neq 0.$$

This means that we have proven one of the inclusions.

Let g be an element of $G^{\mathbb{C}}$ such that $w=g\cdot v$ is not orthogonal to v. The set $P_-K^{\mathbb{C}}P_+$ is open and dense in $G^{\mathbb{C}}$, so we can find sequences $\{p_i^-\}$, $\{k_i\}$ and $\{p_i^+\}$ in P_-, $K^{\mathbb{C}}$ and P_+, respectively, such that

$$g=\lim_{i\to\infty}p_i^-k_ip_i^+.$$

Then

$$w=\lim_{i\to\infty}p_i^-k_ip_i^+\cdot v=\lim_{i\to\infty}p_i^-k_i\cdot v=\lim_{i\to\infty}\lambda(k_i)p_i^-\cdot v.$$

Using the fact that $(p_i^-)^*\cdot v=v$, we see that

$$\lim_{i\to\infty}\lambda(k_i)|v|^2=\lim_{i\to\infty}\langle p_i^-k_i\cdot v,v\rangle\neq 0,$$

so the limit

$$\nu=\lim_{i\to\infty}\lambda(k_i)\in\mathbb{C}$$

exists and is non-zero. Furthermore,

$$w=\nu\lim_{i\to\infty}p_i^-\cdot v.$$

Take elements $Y_i\in\mathfrak{p}_-$ such that $p_i^-=\exp Y_i$. Choose some metric on $\mathfrak{p}_-$ and write $Y_i=z_iZ_i$, where $z_i\in\mathbb{C}$ and $|Z_i|=1$. Then, by passing to some convergent subsequence of $\{Z_i\}$, we can assume that $Z_i\to Z$, for some unit vector $Z\in\mathfrak{p}_-$. As the elements of $\mathfrak{p}_-$ act as nilpotent endomorphisms on V, there is an $n\in\mathbb{N}$ such that $Z_i^n\cdot v=0$ for all i. Thus

$$w=\nu\lim_{i\to\infty}\left(\sum_{k=0}^{n}\frac{z_i^kZ_i^k}{k!}\cdot v\right). \tag{2}$$

If $n = 1$ we must have $Z \cdot v = 0$, which implies that $\exp Z \in P_- \cap P_+ = \{e\}$, so that $Z = 0$, which contradicts the fact that Z is a unit vector. Assume that $n \geqslant 2$. Then

$$\lim_{i\to\infty} Z_i \cdot v + \sum_{k=2}^{n} \frac{z_i^{k-1} Z_i^k}{k!} \cdot v = 0,$$

which implies that

$$\lim_{i\to\infty} \sum_{k=2}^{n} \frac{z_i^{k-1} Z_i^k}{k!} \cdot v = -Z \cdot v.$$

But if $k \geqslant 2$, then

$$\left\langle Z_i^k \cdot v, Z \cdot v\right\rangle = \left\langle Z_i^{k-1} \cdot v, Z_i^* Z \cdot v\right\rangle = \left\langle Z_i^{k-1} \cdot v, [Z^*, Z_i] \cdot v + Z_i Z^* \cdot v\right\rangle = 0,$$

since $Z^* \cdot v = 0$, $[Z^*, Z_i] \cdot v \in [v]$ and $\langle Z_i^{k-1} \cdot v, v\rangle = 0$. Thus,

$$-|Z \cdot v|^2 = \lim_{i\to\infty} \sum_{k=2}^{n} \frac{z_i^{k-1}}{k!} \langle Z_i^k \cdot v, Z \cdot v\rangle = 0,$$

which, as before, is impossible. This shows that the sequence $\{z_i\}$ must be bounded and by passing to a convergent subsequence, we may assume that $z_i \to z$. Thus

$$p_i^- = \exp z_i Z_i \to \exp z Z \in P_-.$$

Hence $w = \nu \exp z Z \cdot v \in P_- \cdot [v]$. □

We are now ready to take the final step in proving Theorem 9.1.

Proof of Lemma 9.2. Following Lemma 9.3 it is sufficient to show that for each $g \in G^{\mathbb{C}}$ we have

$$\left\langle g\sigma^{\mathbb{C}}(g)^{-1} \cdot v, v\right\rangle \neq 0.$$

But this is clear, since by construction, $\sigma^{\mathbb{C}}(g)^{-1} = g^*$, and hence

$$\left\langle g\sigma^{\mathbb{C}}(g)^{-1} \cdot v, v\right\rangle = |g^* v|^2 \neq 0.$$

The statement is proven. □

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